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A Level Further Formula Sheet

Summation Results

Properties (3) (These work similar to integrals)	• Can take constants out: $\sum_{r=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ • Can split up: $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n}(b_k)$ • inclusive exclusive principle $\sum_{k=m}^{n}(a_k) = \sum_{k=1}^{n}(a_k) - \sum_{k=1}^{m-1}(a_k)$
Results (5)	$\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=1}^{n} c = cn$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Matrix Transformations

Reflection in the line $y=(\tan\theta)x$	$\begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$
Horizontal stretch by scale factor k	$\begin{pmatrix}k & 0 \\ 0 & 1\end{pmatrix}$
Vertical stretch by scale factor k	$\begin{pmatrix}1 & 0 \\ 0 & k\end{pmatrix}$
Enlargement by scale factor k centre (0,0)	$\begin{pmatrix}k & 0 \\ 0 & k\end{pmatrix}$
Anti-clockwise rotation of angle θ about origin	$\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \theta > 0$
Clockwise rotation of angle θ about origin	$\begin{pmatrix}\cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}, \theta > 0$

Matrices

Determinant	2×2: $\begin{vmatrix}a & b \\ c & d\end{vmatrix} = ad - bc$ 3×3: $\begin{vmatrix}a & b & c \\ d & e & f \\ g & h & i\end{vmatrix} = a\begin{vmatrix}e & f \\ h & i\end{vmatrix} - b\begin{vmatrix}d & f \\ g & i\end{vmatrix} + c\begin{vmatrix}d & e \\ g & h\end{vmatrix}$ $= a(ei - fh) - b(di - fg) + c(dh - eg)$
Inverse $\frac{1}{determinant}$ $\times$ $adjugate$	2×2: $\begin{pmatrix}a & b \\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -c \\ -b & a\end{pmatrix}$ 3×3: $\begin{pmatrix}a & b & c \\ d & e & f \\ g & h & i\end{pmatrix}^{-1}$ To find adjugate: Step 1: Find matrix of minors (cross of corresponding row and column for each element and the determinant of each remaining part forms the new element) Step 2: Find matrix of cofactors (get the correct signs) $\begin{matrix}+ & - & + \\ - & + & - \\ + & - & +\end{matrix}$ Step 3: Transpose all the elements. In other words swap their positions over the diagonal (the diagonal stays the same)
3 types of solutions for systems of linear equations	• Consistent/unique - one solution (a point) To find unknowns: Use basic elimination or solve det $\neq 0$ (if unknowns on LHS) • Consistent/non-unique – infinite solutions (det = 0) To find unknowns: Use basic elimination with 2 pairs of the same variable eliminated and look to get $0 = 0$ • Inconsistent/non unique – no sol (det = 0) To find unknowns: Use basic elimination with 2 pairs of the same variable eliminated and look to get an inconsistency

Roots

Quadratics, Cubics and Quartics	Quadratic: • form: $(x-\alpha)(x-\beta)$ α and β as the roots • form: $x^2 + bx + c$ sum roots $= -b = \alpha + \beta$ product roots $= c = \alpha\beta$ • form: $ax^2 + bx + c$ sum: $-\frac{b}{a} = \alpha +$, product: $\frac{c}{a} = c = \alpha\beta$ Cubic • form: $(x-\alpha)(x-\beta)(x-\gamma)$ α, β and γ are the roots • form: $ax^3 + bx^2 + cx + d = 0$ sum: $\alpha + \beta + \gamma = -\frac{b}{a}$, product: $\alpha\beta\gamma = -\frac{d}{a}$ sum of all possible products of pairs of roots: $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$ Quartic: • form : $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ α, β, γ and δ are the roots • form: $ax^4 + bx^3 + cx^2 + dx + e = 0$ sum : $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$, product: $\alpha\beta\gamma\delta = \frac{e}{a}$ sum of all possible products of pairs of roots: $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$ sum of all possible products of triples of roots: $\alpha\beta\gamma + \beta\gamma\delta +$ $\gamma\alpha\delta + \delta\alpha\beta = -\frac{d}{a}$
form $\sum_{i=0}^{n} a_i x^i = 0$	Sum $= \frac{-a_{n-1}}{a_n}$, Product $= \frac{(-1)^n a_0}{a_n}$

Polar Coordinates

Area of sector	$\frac{1}{2}\int r^2\, d\theta$

Complex Numbers

Definition	$\sqrt{-1} = i, i^2 = -1$
Cartesian Form	$z = a + bi$
Modulus Argument Form	$z = r(\cos\theta + i\sin\theta) = r\ cis\ \theta$ $r = \sqrt{a^2+b^2}$ $arg = \theta = \tan^{-1}\left(\left\lvert\frac{b}{a}\right\rvert\right)$ Then draw the angle θ in the quadrant where the complex number $a + bi$ lies. Read off θ by starting on the positive x axis (like when you solve for trig using the CAST diagram). Remember that $-\pi \le \theta < \pi$) Or you can use the following to get θ: $\theta = \begin{cases}\tan^{-1}\left(\frac{b}{a}\right) & (if\ quadrant\ 1\ or\ 4) \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi & (if\ quadrant\ 2) \\ \tan^{-1}\left(\frac{b}{a}\right) - \pi & (if\ quadrant\ 3)\end{cases}$
Eulers Form	$z = re^{i\theta}$
De Moivres' Theorem	$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n cis\ n\theta$
Roots of $z^n = 1$	$z = e^{\frac{2\pi k i}{n}}, for\ k = 0,1,2,\ldots,n-1$

Linear Algebra

Eigenvalues	Set characteristic polynomial which is det(A-λI) or det($\lambda - AI$) equal to zero 0, where A = matrix and I= identity matrix. solve this for the eigenvalue
Eigenvectors	**Av**=λ**v** where λ=eigenvector & **v**=eigenvector Av$-\lambda$**v** $= \mathbf{0}$ i.e. A$\begin{pmatrix}x \\ y\end{pmatrix} = \lambda\begin{pmatrix}x \\ y\end{pmatrix}$. Solve with value of λ above Quick way: Put. λ into $(A - \lambda I)$ and you'll obviously have a matrix. Multiply this matrix by $\begin{pmatrix}x \\ y\end{pmatrix}$ and set equal to 0. Solve for x in terms of y

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1-e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2-1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Groups

Order of a group: Number of elements in the group

Order of an element: Least positive integer n such that $g^n = e$ (how many times before you get e in modular arithmetic). If g$\in G$ then g......g=e i.e. $g^n = e$. If no n such that x^n = e then we say element has infinite order

Definition: A set G with a binary operation * on G such that

i. G is associative

ii. G is closed under *

iii. G has an identity element (usually denoted e)

iv. Each element of G has an inverse

Vectors

Notations	$vector = \boldsymbol{a}, \underline{a}, \overrightarrow{OA}$ distance= OA
Vector Form	$a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \equiv \begin{pmatrix}a \\ b \\ c\end{pmatrix}$
Properties (addition/subtraction, multiplication and scalar product)	$\begin{pmatrix}a \\ b \\ c\end{pmatrix} \pm \begin{pmatrix}d \\ e \\ f\end{pmatrix} = \begin{pmatrix}a \pm d \\ b \pm e \\ c \pm f\end{pmatrix}$ $\quad \lambda\begin{pmatrix}a \\ b \\ c\end{pmatrix} = \begin{pmatrix}\lambda a \\ \lambda b \\ \lambda c\end{pmatrix}$ $\begin{pmatrix}a \\ b \\ c\end{pmatrix}.\begin{pmatrix}d \\ e \\ f\end{pmatrix} = ad + be + cf$
Magnitude of a vector Notation is $\lvert\ \rvert$	$\left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix}\right\rvert = \sqrt{a^2 + b^2 + c^2}$
Unit Vector	Unit vector of $\begin{pmatrix}a \\ b \\ c\end{pmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix}a \\ b \\ c\end{pmatrix}$
Parallel and Perpendicular to	Parallel means vectors are a multiple of each other Perpendicular means scalar product equals zero
Angle Between 2 vectors Always use the direction vectors	$\theta = \cos^{-1}\left(\frac{\begin{pmatrix}a \\ b \\ c\end{pmatrix}.\begin{pmatrix}d \\ e \\ f\end{pmatrix}}{\left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}d \\ e \\ f\end{pmatrix}\right\rvert}\right)$
Vector Equation of a line To find this we need: Point and direction (if given 2 points find the directions and use either point)	$r = \begin{pmatrix}a \\ b \\ c\end{pmatrix} + \lambda\begin{pmatrix}d \\ e \\ f\end{pmatrix}$ $\begin{pmatrix}a \\ b \\ c\end{pmatrix} = position, \begin{pmatrix}d \\ e \\ f\end{pmatrix}$ $direction\ (parallel\ to)$
Cartesian Equation of a line	$\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$
Parametric Form of a line	$x = a + \lambda d, y = b + \lambda e, z = c + \lambda f$
Equation of a plane	$\boldsymbol{r}.\boldsymbol{n} = \begin{pmatrix}a \\ b \\ c\end{pmatrix}.\boldsymbol{n}$ where n is the normal vector
Vector Equation of a plane To find this we need: a point in plane and perp direction. If not given perp direction take the cross product of 2 direction vectors. Remember to find a direction we subtract 2 position vectors	$\begin{pmatrix}x \\ y \\ z\end{pmatrix} = \begin{pmatrix}a \\ b \\ c\end{pmatrix} + \lambda\begin{pmatrix}d \\ e \\ f\end{pmatrix} + \mu\begin{pmatrix}r \\ s \\ t\end{pmatrix}$ $\begin{pmatrix}a \\ b \\ c\end{pmatrix} = position$ $\begin{pmatrix}d \\ e \\ f\end{pmatrix}$ and $\begin{pmatrix}r \\ s \\ t\end{pmatrix} = directions\ (parallel\ to)$
Cartesian Equation of a plane	$ax + by + cz = d$ $d = distance\ form\ origin\ to\ plane$ $\begin{pmatrix}a \\ b \\ c\end{pmatrix} = direction\ vector\ (perpendicular\ to)$
Scalar Product Note: θ is the angle between $\begin{pmatrix}a \\ b \\ c\end{pmatrix}$ and $\begin{pmatrix}d \\ e \\ f\end{pmatrix}$	$\begin{pmatrix}a \\ b \\ c\end{pmatrix}.\begin{pmatrix}d \\ e \\ f\end{pmatrix} = \left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}d \\ e \\ f\end{pmatrix}\right\rvert\cos\theta$
Vector Product Note: θ is the angle between $\begin{pmatrix}a \\ b \\ c\end{pmatrix}$ and $\begin{pmatrix}d \\ e \\ f\end{pmatrix}$	$\begin{pmatrix}a \\ b \\ c\end{pmatrix} \times \begin{pmatrix}d \\ e \\ f\end{pmatrix} = \begin{pmatrix}bf - ec \\ -(af - cd) \\ ae - bd\end{pmatrix}$ or $\left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix} \times \begin{pmatrix}d \\ e \\ f\end{pmatrix}\right\rvert = \left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}d \\ e \\ f\end{pmatrix}\right\rvert\sin\theta$
Area of a Parallelogram	$A = \left\lvert\begin{pmatrix}a \\ b \\ c\end{pmatrix} \times \begin{pmatrix}d \\ e \\ f\end{pmatrix}\right\rvert$ $\begin{pmatrix}a \\ b \\ c\end{pmatrix}$ and $\begin{pmatrix}d \\ e \\ f\end{pmatrix}$ form 2 adjacent sides of a parallelogram
Perp Distance between point and plane from (α, β, γ) to $ax + by + cz = d$	$\frac{\lvert a(\alpha) + b(\beta) + c(\gamma) + d\rvert}{\sqrt{a^2 + b^2 + c^2}}$
Scalar Product Properties	$0.\boldsymbol{a} = \boldsymbol{a}$ $\quad \boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$ $(-\boldsymbol{a}).\boldsymbol{b} = -(\boldsymbol{a}.\boldsymbol{b})$ $\quad (k\boldsymbol{a}).\boldsymbol{b} = k(\boldsymbol{a}.\boldsymbol{b})$ $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$ If a and b are parallel: $\boldsymbol{a}.\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert$, moreover $\boldsymbol{a}.\boldsymbol{a} = \lvert\boldsymbol{a}\rvert^2$
Cross Product Properties	$\boldsymbol{a} \times \boldsymbol{a} = 0$ $\quad \boldsymbol{a} \times 0 = 0 \times \boldsymbol{a} = 0$ $\lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda\boldsymbol{b})$ $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$ $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$ $\boldsymbol{b}.(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}.(\boldsymbol{a} \times \boldsymbol{b})$

Trigonometry

If $t = \tan\frac{1}{2}x \Rightarrow \sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

Hyperbolics

Definitions	$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$ $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$ $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$ $\quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
Identities	$\cosh^2 x - \sinh^2 x = 1$ $\quad \tanh^2 x + \operatorname{sech}^2 x = 1$ $\coth^2 x - \operatorname{csch}^2 x = 1$ $\quad \tanh x = \frac{\sinh x}{\cosh x}$ $\sinh 2x = 2\sinh x\cosh x$ $\quad \cosh 2x = \cosh^2 x + \sinh^2 x$
Inverse	$\operatorname{arcosh} x = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \ge 1$ $\operatorname{arsinh} x = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ $\operatorname{artanh} x = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \lvert x\rvert < 1$

Number Theory

Fermat's Theorem	$a^p \equiv a\ (mod\ p)$ if p is prime and a is any integer

Mechanics

Centres Of Mass For Uniform Bodies	• Triangular Lamina: $\frac{2}{3}$ along median from vertex • Circular arc, radius r, angle at centre $2\alpha = \frac{r\sin\alpha}{\alpha}$ from centre • Sector of circle, radius r, angle at centre 2α: $\frac{2r\sin\alpha}{3\alpha}$ from centre • Solid hemisphere, radius r: $\frac{3}{8}r$ from centre • Hemispherical Shell, radius r: $\frac{1}{2}r$ from centre • Solid cone or pyramid of height h: $\frac{1}{4}h$ above the base on the line from centre to base of vertex • Solid cone or pyramid of height h: $\frac{1}{4}h$ above the base on the line from centre to base of vertex • Solid cone or pyramid of height h: $\frac{3}{4}h$ from vertex • Conical shell of height h: $\frac{1}{3}h$ above the base on the line from centre to base of vertex
Motion In A Circle	Transverse velocity: $v = r\dot{\theta}$ Transverse acceleration: $\dot{v} = r\ddot{\theta}$ Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r}$. Note: Mag $= r\dot{\theta}^2$ or $\frac{v^2}{r}$
Motion of a Projectile	Equation of a trajectory: $y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$
Elastic Strings and Springs F =Force needed to extend or compress T=tension x = length of extension/compression k = stiffness constant (spring constant measured in N/m λ = modulus of elasticity (spring modulus) measured in Newtons l =natural length of the spring	Hooke's Law $F = -kx$ Tension in elastic spring/string : $T = \frac{\lambda}{l}x = \frac{\lambda x}{l}$
Energy Note: if answer is negative then means a loss	Kinetic Energy: $\frac{1}{2}mv^2$ Gravitational Potential Energy: mgh Elastic Potential Energy: $\frac{\lambda x^2}{2l}$ Change in kinetic energy: $\frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2$ Change in potential energy: $m_1 g h_1 - m_2 g h_2$
Work Done • W=work done • F=magnitude of the force • d =distance moved IN THE DIRECTION of the force • θ =angle between the force and the displacement • Total energy = Kinetic + Potential + Elastic	If work done against an opposing force: Final total energy=Initial total energy – work done against force where work is done against opposing force is W $W = \begin{cases}Fd \\ d\cos\theta \\ total\ energy\ lost\ (aka\ work\ energy\ principle) \\ potential\ energy\ (if\ talking\ about\ against\ gravity)\end{cases}$ Note: Total energy lost = change in total energy = change in kinetic + change in potential If no work done against an opposing force Final total energy = Initial total energy

Induction Template

Let P_n be the proposition ...

- Let n=1: Plug in n=1 to both the LHS and RHS. Show that LHS=RHS $\Rightarrow P_1$ true
- Assume $n = k$ true i.e. P_k true: Replace n with k. There is nothing to prove here, we just **assume** this to be true.
- Let $n = k + 1$:Replace n with $k + 1$. Usually only need work on the LHS by simplifying and using **assumed P_k** step to show that what we get for LHS is equal to RHS (sometimes we may need to work on the RHS also) $\Rightarrow P_{k+1}$ true

So P_k true $\Rightarrow P_{k+1}$ true

$\because P_1$true then $P_2, P_3, P_4, \ldots$ true

$\therefore true\ for\ all\ n\ \in \cdots$

Types of questions to know: sigma results, divisibility and matrices

Statistics & Probability

Binomial Distribution (discrete)	$x \sim B(n,p)$ E(X)=Mean=np, Var(X)= $np(1-p)$, PMF=$P(X = x)$=$\binom{n}{x}p^x(1-p)^{n-x}$ Calculator: = use pd, $\le$ use cd
Poisson Distribution (discrete) (happening at average rate)	$x \sim Po(\lambda)$ Mean= λ, variance=λ , $PMF = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$ Calculator: =use pd, $\le$ use cd
Uniform Distribution (discrete)	$x \sim U[a,b]$ Mean= $\frac{1}{2}(a+b)$, variance=$\frac{1}{12}(b-a)^2$, PMF=$P(X = x) = \frac{1}{b-a}$ To find unknowns: use fact that area of rectangle is the probability To find probabilities: Find area of rectangle
Geometric (discrete) (how long until 1st success)	$X \sim Geo(p)$ Mean= $\frac{1}{p}$, variance=$\frac{1-p}{p^2}$ PMF=P(X= x)= $p(1-p)^{x-1}$ P(X> r)= $(1-p)^x$ By hand: Need to turn all into = or > and use formulae above Calculator: = use pd, $\le$ use cd
Negative Binomial (discrete) (how long until r successes)	$x \sim NB(r,p)$ $PMF = P(x = k) = \binom{k-1}{r-1}p^r(1-p)^{x-r}$ Mean= $\frac{r}{p}$ variance= $\frac{r(1-p)}{p^2}$
Exponential (waiting time between poisson events)	$x \sim exp(\lambda)$ $PDF = P(x = k) = \lambda e^{-\lambda x}$ Mean= $\frac{1}{\lambda}$ variance= $\frac{1}{\lambda^2}$
Normal Distribution (continuous)	$x \sim N(\mu, \sigma^2)$ Mean= μ, variance= σ^2, $PDF = P(X = x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ Standardised variable $z = \frac{x-\mu}{\sigma}$ To find probabilities: use invnorm on calculator To find x, μ, or σ: use invnorm on calculator
Expected Value Discrete	E(X)= $\sum xP(X = x)$ For a function: $E(g(X)) = \sum g(x)P(X = x)$
Expected Value Continuous	$E(X) = \int xf(x)dx$ For a function: $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$
Variance Discrete	Var(X)= $\sum x^2P(X = x) - E(X)^2 = \sum x^2P(X = x) - \mu^2$
Variance Continuous	$Var\ (X) = \int x^2 f(x)dx - E(X)^2 = \int x^2 f(x)dx - \mu^2$
PDF and CDF	$PDF: f(x_0)$=P(X = x_0) $\quad CDF: F(x_0) = P(X \le x_0) = \int_{-\infty}^{x_0} f(t)dt$ • PDF to CDF i.e. f to F $\Longrightarrow$ integrate (or find areas under graph) Careful with integration if more than 2 functions, always start from the beginning • CDF to PDF i.e.F to f $\Longrightarrow$differentiate (find gradients of graph) • To find probabilities: find areas or integrate $f(x)$ or plug value straight into F(x) • Median=Find m such that $\int_{lower\ limit}^{m} f(x)dx$=0.5 • Mode: Solve $f'(x) = 0$ or if piecewise graph and see which x gives you highest point • Finding unknowns or showing valid PDF:Use fact that $\int f(x)dx$ =1
Probability Generating Function (PGF) PGF is for DRV's only and represents the PMF as a power series i.e. you can find certain (=) probabilities from it Binomial: $(1-p+pt)^n$ Poisson: $e^{\lambda(t-1)}$ Geometric: $\frac{pt}{1-(1-p)t}$ Negative Binomial: $\left(\frac{pt}{1-(1-p)t}\right)^r$	Definition of PGF: $\prod(z)$ or $G(z)$ or $G_x(z)$=$E(z^x)$= $\sum_{k=0\ or\ 1}^{\infty} Z^k P(X = k)$ Note: Sometimes use letter t instead of z Properties: $G(1) = 1$, Mean=E(X)=G'(1) Variance=Var(X)=G''(1)+G'(1)$-(G'(1))^2$ =G''(1)$-\mu(\mu-1)$ If Z= $X + Y$, where X and Y independent: $G_Z(t)$=$G_X(t) \times G_Y(t)$ Finding probabilities: P(X=n)= $\frac{1}{n!}G^{(n)}(0)$ PMF to PGF: Want answer as a power series Fill $P(X = k)$ into $\sum_{k=0\ or\ 1}^{\infty} Z^k P(X = k)$ using PMF and get rid of sigma notation using geometric sum or binomial expan backwards PMF to PGF: PGF should be of the form P(X=0)+zP(X=1)+z^2P(X=2)+...
Moment Generating Function	$M_x(t) = E(e^{tx})$ using either discrete or continuous Expected value def Uniform= $\frac{e^{bt} - e^{at}}{(b-a)t}$, Exponential= $\frac{\lambda}{\lambda - t}$, Normal= $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ Binomial= $[(1-p) + pe^t]^n$, Poisson $= e^{\lambda(e^t - 1)}$, Geometric= $\frac{pe^t}{1-(1-p)e^t}$
Goodness of Fit	${x^2}_{calc} = \sum\frac{(O-E)^2}{E}$. Reject if ${x^2}_{calc} > {x^2}_{critical}$
Spearman's Rank	$1 - \frac{6\sum d^2}{n(n^2-1)}$
Expectation Algebra	$E(aX \pm b) = aE(X) \pm b, VAR(aX \pm b) = a^2Var(X)$ If X and Y independent: $E(XY) = E(X)E(Y), Var(aX \pm bY) = a^2Var(X) + b^2Var(Y)$
Unbiased Estimators	$\overline{x} = \frac{\sum x}{n}$, $s = \sqrt{\frac{\sum x^2}{n-1} - \frac{(\sum x)^2}{n(n-1)}} = \sqrt{\frac{\sum(x - \overline{x})^2}{n-1}} = \sqrt{\frac{n}{n-1}}\sigma$
Central Limit Theorem	$\overline{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
Sample Proportion	$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$
Test Statistics	Z= $\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$, Z = $\frac{(\widehat{p_1} - \widehat{p_2}) - (p_1 - p_2)}{\hat{p}(1-\hat{p})\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$, Z/T = $\frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$, Z = $\frac{\overline{X_1} - \overline{X_2} - (\mu_1 - \mu_2)}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}}$, T= $\frac{\overline{X_1} - \overline{X_2} - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ Where $\widehat{p_1} = \frac{x_1}{n_1}, \widehat{p_2} = \frac{x_2}{n_2}, \hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$ $s_p = \sqrt{\frac{(n_1 - 1){s_1}^2 + (n_2 - 1){s_2}^2}{n_1 + n_2 - 2}}$

Differentiation and Integration

Derivatives	$\sin^{-1} f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-(f(x))^2}}$ $\quad \cos^{-1} f(x) \Rightarrow -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$ $\tan^{-1} f(x) \Rightarrow \frac{f'(x)}{1+(f(x))^2}$ $\quad \sec^{-1} f(x) \Rightarrow \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ $cosec^{-1} f(x) \Rightarrow -\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ $\quad \cot^{-1} f(x) \Rightarrow -\frac{f'(x)}{1+(f(x))^2}$ $\sinh\ f(x) = f'(x)\cosh f(x)$ $\quad \cosh\ f(x) \Rightarrow f'(x)\sinh f(x)$ $\tanh\ f(x) \Rightarrow f'(x)\operatorname{sech}^2 f(x)$ $\quad \sinh^{-1} f(x) \Rightarrow \frac{f'(x)}{\sqrt{1+f(x)^2}}$ $\cosh^{-1} f(x) \Rightarrow \frac{f'(x)}{\sqrt{f(x)^2-1}}$ $\quad \tanh^{-1} f(x) \Rightarrow \frac{f'(x)}{1-f(x)^2}$
Integrals	• $\int\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$ • $\int\frac{1}{-\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + c$ • $\int\frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$ • $\int\frac{1}{\sqrt{a^2+(bx)^2}}\,dx = \frac{1}{b}\sinh^{-1}\left(\frac{bx}{a}\right) + c = \frac{1}{b}\ln\left(bx + \sqrt{(bx)^2 + a^2}\right) + c$ • $\int\frac{1}{\sqrt{(bx)^2-a^2}}\,dx = \frac{1}{b}\cosh^{-1}\left(\frac{bx}{a}\right) + c = \frac{1}{b}\ln\left(bx + \sqrt{(bx)^2 - a^2}\right) + c$, x>a • $\int\frac{1}{a^2-(bx)^2}\,dx = \frac{1}{ab}\tanh^{-1}\left(\frac{bx}{a}\right) + c = \frac{1}{2ab}\ln\left\lvert\frac{a+bx}{a-bx}\right\rvert + c$, $\lvert x\rvert < 1$ • $\int\frac{1}{(bx)^2-a^2}\,dx = \frac{1}{2ab}\ln\left\lvert\frac{bx-a}{bx+a}\right\rvert + c$
Volume of Revolution	About $x\ axis$: $V = \int_a^b \pi y^2 dx$, About y axis: $V = \int_a^b \pi x^2 dy$
Surface area of revolution	Cartesian: $2\pi\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ Parametric: $2\pi\int y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ Polar: $2\pi\int r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$
Arc Length	Cartesian: $\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ Parametric: $\int\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ Polar: $\int\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Differential Equations

Integrating Factor	$y' + P(x)y = Q(x) \Longrightarrow Integ\ Factor = e^{\int P(x)dx}$
Homogeneous and Non-Homogeneous (Second Order) Solution Form: $y = y_{cf} + y_p$	complementary function y_{cf} (set LHS 0) • real roots: $m_1, m_2 \Rightarrow y_{cf} = Ae^{m_1 x} + Be^{m_2 x}$ $= e^{px}[A\cosh(mx) + A\sinh(mx)]$ if roots are $p \pm m$ • repeated roots: $m \Rightarrow y_{cf}$= $Ae^{mx} + Bxe^{mx} = e^{mx}(A + Bx)$ • complex roots: $p + iq \Rightarrow y_{cf} = e^{px}(A\cos qx + B\sin qx)$ Non-homogeneous y_p (plug into RHS of equation & solve for unknowns) • $y = k$ (constant) $\Longrightarrow$ use $y = k$ • $y = x \Longrightarrow y = p + qx$ • $y = ax^n \Longrightarrow y = Ax^n + Bx^{n-1} + \cdots + Mx + N$ • $y = ae^{bx} \Longrightarrow y = ce^{bx}$ $y = cxe^{bx}$ (if b matches root of CF) $y = cx^2e^{bx}$ (if b matches repeated root of CF) • $y = a\cos bx\ or\ a\sin bx$ or y =any linear combo of $\sin$ or $\cos$ $\Longrightarrow y = D\cos bx + E\sin bx$ $y = x(D\cos bx + E\sin bx)$ if b matches repeated root of CF)
Maclaurin Series	$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$
Taylor's Series	$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$
Maclaurin and Taylors Series Common Results (5)	• $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ for all x • $\ln(1+x) = x - \frac{x^2}{2} + \frac{3}{3} + \cdots$ for $-1 < x \le 1$ • $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^r\frac{x^{2r+1}}{(2r+1)!}$ for all x • $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^r\frac{x^{2r}}{(2r)!}$ for all x • $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^r\frac{x^{2r+1}}{(2r+1)!}$ for $-1 \le x \le 1$ • $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2r+1}}{(2r+1)!}$ for all x • $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2r}}{(2r)!}$ for all x • $\tanh x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2r+1}}{(2r+1)!}$ for $-1 \le x \le 1$